Introduction and Basics in TOH

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Introduction

- The objective of statistics is to make inference about a population based on information contained in the sample.
- **Population** denotes all items the researcher is interested in.
- **Sample** A subset of the population
- There are two major areas of statistical inference namely:
  i. Estimation of parameters
  ii. Testing of hypothesis
- We will develop general methods for testing hypothesis and apply to some common problems.

Basics

- We will consider a random variable X with pdf $f(x;\theta)$ where θ is an unknown parameter.
- **Statistical hypothesis** is a statement or assertion about the distribution of one or more random variables.
- **Hypothesis testing** is a rule for deciding whether to reject or accept the statistical hypothesis.
- **Null Hypothesis** This represents a theory that has been put forward either because it is believed to be true or because it is used as a basis for argument but has not been proved. Denoted by H_0
- **Alternative Hypothesis** denoted by H_1 means an opposing theory to the null hypothesis.

Basics Cont'd

- The final conclusion once the test have been carried out is always given in terms of the null hypothesis.
- The two possible conclusions are:
  - i. Reject H_0
  - ii. Fail to reject H_0

Simple and Composite Hypothesis

- A hypothesis is said to be **simple** if it specifies the population completely *i.e it specifies the probability distribution uniquely* whereas a **composite** hypothesis leads to two or more possibilities.
- Examples:
  i. Suppose $X \sim N(\mu, 1)$

$$H_0 : \mu = 0 \rightarrow X \sim N(0, 1)$$

$$H_a : \mu = 2 \rightarrow X \sim N(0, 2)$$

Both H_0 and H_1 are simple hypothesis.

ii. Suppose $X \sim N(\mu, \sigma^2)$

$$H_0 : \mu = 5, \sigma^2 = 25 \rightarrow X \sim N(5, 25)$$

$$H_1 : \mu \neq 5, \sigma^2 = 6$$

H_0 is simple while H_1 is composite.

iii. Suppose $X \sim N(\mu, \sigma^2)$

$$H_0 : \mu \leq 25, \sigma^2 \leq 9$$

$$H_a : \mu > 25, \sigma^2 > 9$$

Both H_0 and H_1 are composite since we do not know the exact values of μ and σ^2

Rejection and Acceptance Regions

- Let $X_1, X_2, ..., X_n$ be a random sample of size n from X. These observations generate a n-dimensional space which we denote by X, a sample space in $\mathbb{R}^n$
- A statistical test partitions the sample space X into two disjoint subsets ω and $\bar{\omega}$ that if
  a. $x \in \omega$ we reject H_0
  b. $x \in \bar{\omega}$ we accept H_0 where $X = (x_1, x_2, ..., x_n)$ is a particular value of the observed vector.
- We call ω the rejection or critical region and $\bar{\omega}$ is called the acceptance region.
- Acceptance region is the set of values for which the null hypothesis is accepted while rejection(critical) region is the set of values for the test statistic for which H_0 is rejected.

Errors in decisions

- There are 2 possible errors:
  i. Type I error: Reject H_0 if it is true
  ii. Type II error: Accepting H_0 if it is false

	H_0 True	H_0 False
Reject H_0	Type I Error	Correct Rejection
Fail to Reject H_0	Correct Decision	Type II Error

- $\alpha = P(Type \;\; I \;\; error)$
- $\beta = P(Type \;\; II \;\; error)$
- $1 - \beta = Power \;\; of \;\; test$

Significance level of a test

- The significance level of a statistical hypothesis test is a fixed probability of wrongly rejecting the null hypothesis H_0 if infact H_0 is true.
- It is the probability of type I error and is set by the investigator in relation to the consequences of such an error.
- The significance level is usually denoted by α thus $\alpha =$ P(Type I error)
- Usually the level of significance is chosen to be $0.001 = 0.1\%, 0.01 = 1\%, 0.05 = 5\%, 0.10 = 10\%$

P-Value

- The probability value, p-value of a statistical hypothesis is the probability of getting a value of the test statistic at least as extreme as that observed by chance alone if the null hypothesis is true.
- It is the probability of wrongly rejecting null hypothesis if it true.
- We can make a decision on whether to reject or fail to reject H_0 based on the p-value.
- If the p-value is less than 0.05 α level of significance we reject H_0.

Example 1

Suppose a researcher is to test the hypothesis $\theta = 0.90$ against the alternative $\theta = 0.60$. Let this test statistic be X, the observed number of successes in $n = 20$ trials. If he will accept the hypothesis if $X > 15$ or otherwise conclude $\theta = 0.60$. Obtain the significance level of the test and the probability of obtaining type II error.

Solution

- We have the hypothesis

$$H_0 : \theta = 0.90$$

vs

$$H_a = 0.60$$

- The critical region is

$$\omega = \{0 \leq X \leq 14\}$$

- The acceptance region is

$$\bar{\omega} = \{15 \leq X \leq 20\}$$

Cont'd

- The distribution is:

$$f(x,\theta) = \binom{20}{x}\theta^x(1-\theta)^{20-x}, \quad x = 0, 1, 2...20 \tag{1}$$

$$\begin{aligned}\alpha &= P(TypeIerror)\\ &= P(X \in \omega | H_0 is \quad true)\\ &= P(X \leq 15 | \theta = 0.90)\end{aligned} \tag{2}$$

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$$\begin{aligned}f(x,\theta) &== \binom{20}{x}(0.9)^x(0.1)^{20-x}\\ &= \sum_{x=0}^{14}\binom{20}{x}(0.9)^x(0.1)^{20-x} = 0.0114\end{aligned} \tag{3}$$

Cont'd

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$$\beta = P(X \in \bar{\omega} | H_0 is \quad false) \\ = P(x \in \bar{\omega} | H_1 is \quad true) \\ = P(X \in \bar{\omega} | \theta = 0.60) \tag{4}$$

$$f(x, \theta) == \binom{20}{x} (0.6)^x (0.4)^{20-x} \\ = \sum_{x=15}^{20} \binom{20}{x} (0.9)^x (0.1)^{20-x} = 0.1256 \tag{5}$$

Example 2

If $X \geq 1$ is the critical region for testing $\theta = 0.2$ against $\theta = 1$ on the basis of a single observation from the population whose pdf is

$$f(x, \theta) = \begin{cases} \theta e^{-\theta x} & if 0 \leq X < \theta \\ 0 & otherwise \end{cases} \tag{6}$$

Obtain the values of Type I and Type II errors.

Solution

- $$H_0 : \theta = 2 \;\; vs \;\; H_1 : \theta = 1$$

- $$\omega = \{X : X \geq 1\} \;\; \bar{\omega} = \{X : X < 1\}$$

- Then

$$\begin{aligned} \alpha &= P(Reject \;\; H_0 | H_0 \;\; is \;\; true) \\ &= P(x \in \omega | \omega = 2) \\ &= P(1 \leq X < 2 | \theta = 2) \end{aligned} \tag{7}$$

Cont'd

- $$\begin{aligned} f(x,\theta=2) &= 2e^{-2x} \\ &= \int_1^\infty 2e^{-2x}dx \\ &= -e^{-2x}]_1^\infty \\ &= 0 + e^{-2} = 0.1353 \end{aligned} \tag{8}$$

- $$\begin{aligned} \beta &= P(accept\ \ H_0|H_1\ \ is\ \ True) \\ &= P(x \in \bar{\omega}|\theta = 1) \\ &= P(X < 1|\theta = 1) \\ &= f(x,\theta = 1) = e^{-x}, x \geq 0 \\ &= \int_0^1 e^{-x}dx = -e^{-x}]_0^1 = 0.6321 \end{aligned} \tag{9}$$

Power of a test

- The power of a test is the probability of rejecting the null hypothesis when it is false.
- It is given by $1 - \beta$ where β is the probability of type II error.
- It is therefore the probability of not commiting type II error.
- The power function of a statistical test of hypothesis H_0 against an alternative H_1 is given by:

$$\pi(\theta) = \alpha(\theta)$$

for values of θ under H_0

$$\pi(\theta) = 1 - \beta(\theta)$$

for values of θ under H_1

Example

Let p be the probability that a coin will fall as a head in a single toss in order to test

$$H_0 : p = \frac{1}{2} \;\; vs \;\; H_1 : p = \frac{3}{4}$$

The coin is tossed 5 times and H_0 is rejected if more than 3 heads are obtained. Find the power of the test.

Solution

- X forms a binomial distribution with parameters $n = 5$ and $p = 0.5$ thus

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{5-x}, x = 0, 1, 2, 3, 4, 5$$

- The Rejection region $\omega = \{x : x > 3\}$ and $\bar{\omega} = \{x : x \leq 3\}$
- $\beta = P(x \in \bar{\omega} | H_1 is \ \ true) = P(X \leq 3 | p = \frac{3}{4})$
- $$\sum_{x=0}^{3} \binom{5}{x} (\frac{3}{4})^x (\frac{1}{4})^{5-x} = 0.3671$$

  .
- The power of the test is $1 - \beta = 1 - 0.3671 = 0.6329$

Thank You!